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Symmetries of Differential Equations and Cartan's Equivalence Method

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Applications of Cartan's equivalence method to symmetries of differential equations are considered. The examples include interrelations between the nonlinear acoustics equations, and the solutions of equivalence problems for the classes of linear parabolic equations and nonlinear wave equations.

Introduction

In this work we consider applications of Élie Cartan's structure theory of Lie pseudo-groups to symmetry groups of differential equations. This approach allows to solve equivalence problems for classes of differential equations and to find contact transformations mapping equivalent equations into each other. Also, it allows to obtain all differential invariants of symmetry groups for differential equations without analysis of over-determined systems of partial differential equations.

Definition 1. [20, 13] A *pseudo-group* $\mathcal{G}$ on a manifold M is a collection of local diffeomorphisms of M, which is closed under composition *when defined*, contains an identity and is closed under inverse. A *Lie pseudo-group* is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations.

Example 1. The pseudo-group of conformal transformations in $\mathbb{R}^2$ consists of local mappings $(x, y) \mapsto (X, Y)$ satisfying the Cauchy - Riemann equations $X_x = Y_y$, $X_y = -Y_x$.

É. Cartan developed an approach to study Lie pseudo-groups by means of exterior differential forms. He showed that for every Lie pseudo-group $\mathcal{G}$ acting on a manifold M there exists a finite-dimensional Lie group H and a set of 1-forms $\Omega = \{\omega^1, ..., \omega^s\}$ on a direct product $M \times H$ such that a transformation $\Delta : M \to M$ belongs to $\mathcal{G}$ if and only if the forms Ω are invariant under a lift $\hat{\Delta} : M \times H \to M \times H$. The forms Ω are called a *moving coframe* or *tautological forms* for the pseudo-group $\mathcal{G}$.

Example 1 (continued). For the pseudo-group of conformal transformations in $\mathbb{R}^2$ we can take the moving coframe $\omega^1 = a\,dx - b\,dy$, $\omega^2 = b\,dx + a\,dy$ on $\mathbb{R}^2 \times H$, where H is a Lie group of non-degenerate matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Taking differentials of the tautological forms, we obtain Cartan's *structure equations* for the pseudo-group

$$d\omega^i = A^i_{\alpha k}\, \eta^\alpha \wedge \omega^k + T^i_{jk}\, \omega^j \wedge \omega^k,$$

where $A^i_{\alpha k}$, as usual, are constants, the *torsion coefficients* T^i_{jk} are invariants of the pseudo-group, and η^α are linear combinations of Maurer-Cartan forms for the group H and the forms Ω. Cartan gave an explicit, practical test for involutivity of moving coframes. The test deals with ranks of matrices constructed from $A^i_{\alpha k}$. For non-involutive coframes Cartan's prolongation procedure allows us to obtain an involutive coframe after a finite number of extensions of the group H.

Example 1 (continued). The structure equations for the pseudo-group of conformal transformations are $d\omega^1 = \pi_1 \wedge \omega^1 - \pi^2 \wedge \omega^2$, $d\omega^2 = \pi_2 \wedge \omega^1 + \pi^1 \wedge \omega^2$, where $\pi_1 = (a\,da + b\,db)\,(a^2 + b^2)^{-1} + t_1\,\omega^1 + t_2\,\omega^2$, $\pi_2 = (-b\,da + a\,db)\,(a^2 + b^2)^{-1} - t_2\,\omega^1 + t_1\,\omega^2$, while t_1 and t_2 are free parameters.

Full details of Cartan's method can be found in [1, 2, 3, 17].

1 Contact transformations

Using Cartan's method, we can find a moving coframe for the pseudo-group $Cont(J^1(\mathcal{E}))$ of contact transformations on a bundle $J^1(\mathcal{E})$ of first-order jets of sections of $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. In the local coordinates $(x^i, u^\alpha, p^\alpha_i)$, $i \in \{1, ..., n\}$, $\alpha \in \{1, ..., m\}$, a contact transformation $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$, $\Delta : (x^i, u^\alpha, p^\alpha_i) \mapsto (\overline{x}^i, \overline{u}^\alpha, \overline{p}^\alpha_i)$, is defined by the requirement to map the *contact forms* $\vartheta^\alpha = du^\alpha - p^\alpha_i\,dx^i$ into linear combinations of the contact forms: $\Delta^*\overline{\vartheta}^\alpha = d\overline{u}^\alpha - \overline{p}^\alpha_i\,d\overline{x}^i = \zeta^\alpha_\beta(x, u, p)\,\vartheta^\beta$ for appropriate functions ζ^α_β on $J^1(\mathcal{E})$.

As it is shown in [14], we can take the following moving coframe for $Cont(J^1(\mathcal{E}))$:

$$\Theta^\alpha = a^\alpha_\beta\,(du^\beta - p^\beta_j\,dx^j), \quad \Xi^i = c^i_\beta\,\Theta^\beta + b^i_j\,dx^j, \quad \Sigma^\alpha_i = f^\alpha_{i\beta}\,\Theta^\beta + g^\alpha_{ij}\,\Xi^j + a^\alpha_\beta\,B^j_i\ dp^\beta_j. \qquad (1)$$

These forms are defined on $J^1(\mathcal{E}) \times H$, where H is the Lie group of block lower triangular matrices

$$\left(\begin{array}{ccc} a^\alpha_\beta & 0 & 0 \\ c^i_\gamma\,a^\gamma_\beta & b^i_j & 0 \\ (f^\alpha_{i\gamma} + g^\alpha_{ik}\,c^k_\gamma)\,a^\gamma_\beta & g^\alpha_{ik}\,b^k_j & a^\alpha_\beta\,B^j_i \end{array}\right),$$

and the parameters a^α_β, b^i_j, c^i_β, $f^\alpha_{i\beta}$, and g^α_{ij} obey the requirements $\det\left(a^\alpha_\beta\right) \neq 0$, $\det\left(b^i_j\right) \neq 0$, $b^i_k\,B^k_j = \delta^i_j$, and $g^\alpha_{ij} = g^\alpha_{ji}$. It is easy to verify directly that Δ is a contact transformation whenever (1) is invariant under an action of a lift of Δ on $J^1(\mathcal{E}) \times H$. The structure equations for the lifted coframe (1) have the form

$$\begin{aligned} & d\Theta^\alpha = \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k, \quad d\Xi^i = \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma, \\ & d\Sigma^\alpha_i = \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Upsilon^\alpha_{ij} \wedge \Xi^j, \end{aligned} \qquad (2)$$

where Φ^α_β, Ψ^i_k, Π^i_γ, $\Lambda^\alpha_{i\beta}$, and Υ^α_{ij} are appropriate 1-forms on $J^1(\mathcal{E}) \times H$. From these equations it follows that the moving coframe (1) is involutive. The structure equations (2) remain unchanged if we make the following change of the forms Φ^α_β, Ψ^i_k, Π^i_γ, $\Lambda^\alpha_{i\beta}$, and Υ^α_{ij} :

$$\begin{aligned} & \Phi^\alpha_\beta \mapsto \Phi^\alpha_\beta + K^\alpha_{\beta\gamma}\,\Theta^\gamma, \quad \Psi^i_k \mapsto \Psi^i_k + L^i_{kj}\,\Xi^j + M^i_{k\gamma}\,\Theta^\gamma, \quad \Pi^i_\gamma \mapsto \Pi^i_\gamma + M^i_{k\gamma}\,\Xi^k + N^i_{\gamma\epsilon}\,\Theta^\epsilon, \\ & \Lambda^\alpha_{i\beta} \mapsto \Lambda^\alpha_{i\beta} + P^\alpha_{i\beta\gamma}\,\Theta^\gamma + Q^\alpha_{i\beta k}\,\Xi^k + K^\alpha_{\gamma\beta}\,\Sigma^\gamma_i - M^k_{i\beta}\,\Sigma^\alpha_k, \\ & \Upsilon^\alpha_{ij} \mapsto \Upsilon^\alpha_{ij} + Q^\alpha_{i\beta j}\,\Theta^\beta + R^\alpha_{ijk}\,\Xi^k - L^k_{ij}\,\Sigma^\alpha_k, \end{aligned} \qquad (3)$$

where $K^\alpha_{\beta\gamma}$, L^i_{kj}, $M^i_{k\gamma}$, $N^i_{\gamma\epsilon}$, $P^\alpha_{i\beta\gamma}$, $Q^\alpha_{i\beta k}$, and R^α_{ijk} are arbitrary functions on $J^1(\mathcal{E}) \times H$ satisfying the symmetry conditions $K^\alpha_{\beta\gamma} = K^\alpha_{\beta\gamma}$, $L^i_{kj} = L^i_{jk}$, $N^i_{\gamma\epsilon} = N^i_{\epsilon\gamma}$, $P^\alpha_{i\beta\gamma} = P^\alpha_{i\gamma\beta}$, $Q^\alpha_{i\beta k} = Q^\alpha_{k\beta i}$, $R^\alpha_{ijk} = R^\alpha_{ikj} = R^\alpha_{jik}$.

Another approach to construct 1-forms characterizing contact transformations is presented in [18].

2 Symmetries of differential equations

Symmetry groups of differential equations are sub-groups of pseudo-groups of contact transformations. Therefore Cartan's method of equivalence is applicable to find tautological forms for these symmetry groups. We briefly outline this approach, [4, 14].

Every partial differential equation (PDE) can be transformed into an equivalent PDE of the first order by adding appropriate new variables, [20, th 3.3.1], [6, § 17.4]. Therefore we take a PDE $\mathcal{R}$ of the first order. $\mathcal{R}$ is a sub-bundle in $J^1(\mathcal{E})$. Let $Sym(\mathcal{R})$ be the group of contact symmetries for $\mathcal{R}$. It consists of all the contact transformations on $J^1(\mathcal{E})$ mapping $\mathcal{R}$ to itself. Let $\iota : \mathcal{R} \to J^1(\mathcal{E})$ be an embedding. The tautological forms of $Sym(\mathcal{R})$ are restrictions of the lifted coframe (1) on $\mathcal{R}$: $\theta^\alpha = \iota^*\Theta^\alpha$, $\xi^i = \iota^*\Xi^i$, and $\sigma^\alpha_i = \iota^*\Sigma^\alpha_i$ (for brevity we identify the map $\iota \times id : \mathcal{R} \times H \to J^1(\mathcal{E}) \times H$ with $\iota : \mathcal{R} \to J^1(\mathcal{E})$). The forms θ^α, ξ^i, and σ^α_i have some linear dependencies, i.e., there exists a non-trivial set of functions S_α, T_i, and U^i_α on $\mathcal{R} \times H$ such that $S_\alpha\, \theta^\alpha + T_i\, \xi^i + U^i_\alpha\, \sigma^\alpha_i \equiv 0$. These functions are lifted invariants of $Sym(\mathcal{R})$. Setting them equal to appropriate constants allows us to specify some parameters a^α_β, b^i_j, c^i_β, $f^\alpha_{i\beta}$, and g^α_{ij} of the group H as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

After these normalizations, some restrictions of the forms $\phi^\alpha_\beta = \iota^*\Phi^\alpha_\beta$, $\psi^i_k = \iota^*\Psi^i_k$, and $\pi^i_\beta = \iota^*\Pi^i_\beta$, or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of H. From (3), we have the following statements: (i) if ϕ^α_β is semi-basic, then its coefficients at σ^γ_j and ξ^j are lifted invariants of $Sym(\mathcal{R})$; (ii) if ψ^i_k or π^i_β are semi-basic, then their coefficients at σ^γ_j are lifted invariants of $Sym(\mathcal{R})$. Setting these invariants equal to some constants, we get specifications of some more parameters of H as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

$$
\begin{aligned}
&d\theta^\alpha = \phi^\alpha_\beta \wedge \theta^\beta + \xi^k \wedge \sigma^\alpha_k, \quad d\xi^i = \psi^i_k \wedge \xi^k + \pi^i_\gamma \wedge \theta^\gamma,\\
&d\sigma^\alpha_i = \phi^\alpha_\gamma \wedge \sigma^\gamma_i - \psi^k_i \wedge \sigma^\alpha_k + \lambda^\alpha_{i\beta} \wedge \theta^\beta + \upsilon^\alpha_{ij} \wedge \xi^j.
\end{aligned}
$$

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of tautological forms for $Sym(\mathcal{R})$. In the second case, when the reduced lifted coframe does not satisfy Cartan's test, we should use the procedure of prolongation, [17, ch 12].

Another approach to find structure equations, but not tautological forms, for symmetry groups of PDEs is given in [12, 13].

3 Interrelations between the Khokhlov-Zabolotskaya equation and the short wave gas dynamics equation

Consider two equations: the Khokhlov - Zabolotskaya equation, [23, ch 3, § 5.3], [21], in the potential form

$$
v_x = u_y, \qquad v_y = u_t + u\, u_x, \tag{4}
$$

and the short wave gas dynamics equation, [6, § 23.4], [5],

$$
v_x = u_y, \qquad v_y = u_t + (u + x)\, u_x + k\, u. \tag{5}
$$

The moving coframe method gives the following structure equations for the symmetry pseudo-group of (4):

$$
\begin{aligned}
d\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \sigma^1_1 + \xi^2 \wedge \sigma^1_2 + \xi^3 \wedge \sigma^1_3,\\
d\theta^2 &= \frac{3}{2}\,\eta_1 \wedge \theta^1 + \eta_3 \wedge \theta^1 + \xi^1 \wedge \sigma^2_1 + \xi^2 \wedge \sigma^1_3 + \xi^3 \wedge \sigma^1_1 + \xi^3 \wedge \sigma^1_2,\\
d\xi^1 &= \eta_2 \wedge \xi^1,\\
d\xi^2 &= -\eta_1 \wedge \xi^1 + \eta_1 \wedge \xi^2 + \eta_2 \wedge \xi^2 - \eta_3 \wedge \xi^3 - \theta^1 \wedge \xi^1,\\
d\xi^3 &= \frac{1}{2}\,\eta_1 \wedge \xi^3 + \eta_2 \wedge \xi^3 - 2\,\eta_3 \wedge \xi^1,\\
d\sigma^1_1 &= \eta_1 \wedge \left(\sigma^1_1 + \sigma^1_2\right) - \eta_2 \wedge \sigma^1_1 + 2\,\eta_3 \wedge \sigma^1_3 + \eta_4 \wedge \xi^3 + \eta_5 \wedge \xi^2 + \eta_6 \wedge \xi^1 - \theta^1 \wedge \sigma^1_2,\\
d\sigma^1_2 &= -\eta_2 \wedge \sigma^1_2 + \eta_5 \wedge \xi^1,\\
d\sigma^1_3 &= \frac{1}{2}\,\eta_1 \wedge \sigma^1_3 - \eta_2 \wedge \sigma^1_3 + \eta_3 \wedge \sigma^1_2 + \eta_4 \wedge \xi^1 + \eta_5 \wedge \xi^3,\\
d\sigma^2_1 &= \eta_1 \wedge \left(\sigma^1_3 + \frac{1}{2}\,\sigma^2_1\right) - \eta_2 \wedge \sigma^2_1 + \eta_3 \wedge \left(3\,\sigma^1_1 + 2\,\sigma^1_2\right) + \eta_4 \wedge \xi^2 + \eta_5 \wedge \xi^3 + \eta_6 \wedge \xi^3\\
&\quad + \eta_7 \wedge \xi^1 - 3\,\theta^2 \wedge \sigma^1_2,\\
d\eta_1 &= 2\,\xi^1 \wedge \sigma^1_2,\\
d\eta_2 &= -3\,\xi^1 \wedge \sigma^1_2,\\
d\eta_3 &= \frac{1}{2}\,\eta_1 \wedge \eta_3 + \xi^1 \wedge \sigma^1_3 + \xi^3 \wedge \sigma^1_2,\\
d\eta_4 &= \pi_1 \wedge \xi^1 + \pi_2 \wedge \xi^3 + \frac{1}{2}\,\eta_1 \wedge \eta_4 - 2\,\eta_2 \wedge \eta_4 + 3\,\eta_3 \wedge \eta_5 - 3\sigma^1_2 \wedge \sigma^1_3,\\
d\eta_5 &= \pi_2 \wedge \xi^1 - 2\,\eta_2 \wedge \eta_5,\\
d\eta_6 &= \pi_1 \wedge \xi^3 + \pi_2 \wedge \xi^2 + \pi_3 \wedge \xi^1 + 2\,\eta_1 \wedge \eta_5 + \eta_1 \wedge \eta_6 - 2\,\eta_2 \wedge \eta_6 + 4\,\eta_3 \wedge \eta_4\\
&\quad + 6\,\sigma^1_1 \wedge \sigma^1_2,\\
d\eta_7 &= \pi_1 \wedge \xi^2 + \pi_2 \wedge \xi^3 + \pi_3 \wedge \xi^3 + \pi_4 \wedge \xi^1 + 2\,\eta_1 \wedge \eta_4 + \frac{3}{2}\,\eta_1 \wedge \eta_7 - 2\,\eta_2 \wedge \eta_7 + 4\,\eta_3 \wedge \eta_5\\
&\quad + 5\,\eta_3 \wedge \eta_6 - \eta_4 \wedge \theta^1 + 3\,\eta_5 \wedge \theta^2 + 3\,\sigma^1_1 \wedge \sigma^1_3 - 9\,\sigma^1_2 \wedge \sigma^2_1.
\end{aligned}
\tag{6}
$$

The structure of symmetry pseudo-group of the equation (5) depends on the value of the parameter k. Particularly, if $k = -2$ or $k = -\frac{1}{2}$, the pseudo-group has exactly the same structure equations (6) but with different 1-forms θ^1, θ^2, ξ^1, ξ^2, ξ^3, σ^1_1, σ^1_2, σ^1_3, σ^2_1, η_1, ... , η_7, π_1, ... , π_4. Therefore, in the cases $k = -2$ or $k = -\frac{1}{2}$ there exist transformations mapping equation (5) into equation (4).

For the other values of k the structure equations for the symmetry pseudo-group of equation (5) have the form

$$
\begin{aligned}
d\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \sigma^1_1 + \xi^2 \wedge \sigma^1_2 + \xi^3 \wedge \sigma^1_3,\\
d\theta^2 &= \frac{3}{2}\,\eta_1 \wedge \theta^1 + \eta_2 \wedge \theta^1 + \xi^1 \wedge \sigma^2_1 + \xi^2 \wedge \sigma^1_3 + \xi^3 \wedge \sigma^1_1 + \xi^3 \wedge \sigma^1_2,\\
d\xi^1 &= 0,\\
d\xi^2 &= \eta_1 \wedge \left(\xi^2 - \xi^1\right) - \eta_2 \wedge \xi^3 - \theta^1 \wedge \xi^1 + E\,\xi^1 \wedge \xi^2,\\
d\xi^3 &= \frac{1}{2}\,\eta_1 \wedge \xi^3 - 2\,\eta_2 \wedge \xi^1 + E\,\xi^1 \wedge \xi^3,\\
d\sigma^1_1 &= \eta_1 \wedge \left(\sigma^1_1 + \sigma^1_2\right) + 2\,\eta_2 \wedge \sigma^1_3 + \eta_3 \wedge \xi^3 + \eta_4 \wedge \xi^1 - \theta^1 \wedge \sigma^1_2 + E\,\xi^2 \wedge \sigma^1_2,\\
d\sigma^1_2 &= 0,
\end{aligned}
$$

$$
\begin{aligned}
d\sigma^1_3 &= \frac{1}{2}\,\eta_1\wedge\sigma^1_3 + \eta_2\wedge\sigma^1_2 + \eta_3\wedge\xi^1 - E\,\xi^1\wedge\sigma^1_3 + E\,\xi^3\wedge\sigma^1_2,\\
d\sigma^2_1 &= \eta_1\wedge\sigma^1_3 + \frac{3}{2}\,\eta_1\wedge\sigma^2_1 + 3\,\eta_2\wedge\sigma^1_1 + 2\,\eta_2\wedge\sigma^1_2 + \eta_3\wedge\xi^2 + \eta_4\wedge\xi^3 + \eta_5\wedge\xi^1\\
&\quad + \frac{1}{9}\,\theta_1\wedge\xi^3 - 3\,\theta_2\wedge\sigma^1_2 + E\,\xi^3\wedge\left(\sigma^1_1+\sigma^1_2\right),\\
d\eta_1 &= 2\,\xi^1\wedge\sigma^1_2,\\
d\eta_2 &= \frac{1}{2}\,\eta_1\wedge\eta_2 - \frac{1}{9}\,\xi^1\wedge\xi^3 + \xi^1\wedge\sigma^1_3 + \xi^3\wedge\sigma^1_2,\\
d\eta_3 &= \pi_1\wedge\xi^1 + \frac{1}{2}\,\eta_1\wedge\eta_3 - 3\,E\,\eta_2\wedge\sigma^1_2 + \left(\frac{1}{9} - 2\,E^2\right)\xi^3\wedge\sigma^1_2 - 3\,\sigma^1_2\wedge\sigma^1_3,\\
d\eta_4 &= \pi_1\wedge\xi^3 + \pi_2\wedge\xi^1 + \eta_1\wedge\eta_4 - E\,\eta_1\wedge\sigma^1_2 + 4\,\eta_2\wedge\eta_3 + 2\,E\,\eta_2\wedge\sigma^1_3 - E\,\eta_3\wedge\xi^3\\
&\quad - E\,\theta^1\wedge\sigma^1_2 - E^2\,\xi^2\wedge\sigma^1_2 + \frac{2}{9}\,\xi^3\wedge\sigma^1_3 + 3\,\sigma^1_1\wedge\sigma^1_2\\
d\eta_5 &= \pi_1\wedge\xi^2 + \pi_2\wedge\xi^3 + \pi_3\wedge\xi^1 + 2\,\eta_1\wedge\eta_3 + \frac{3}{2}\,\eta_1\wedge\eta_5 + E\,\eta_1\wedge\sigma^1_3 + 5\,\eta_2\wedge\eta_4\\
&\quad + \frac{2}{9}\,\eta_2\wedge\theta^1 - 2\,E\,\eta_2\wedge\left(\sigma^1_1+\sigma^1_2\right) - \eta_3\wedge\theta^1 - E\,\eta_3\wedge\xi^2 - 2\,E\,\eta_4\wedge\xi^3 - \frac{1}{9}\,E\,\theta^1\wedge\xi^3\\
&\quad + \left(\frac{4}{9} - E^2\right)\xi^3\wedge\sigma^1_1 + \left(\frac{2}{9} - E^2\right)\xi^3\wedge\sigma^1_2 + 3\,\sigma^1_1\wedge\sigma^1_3 - 6\,\sigma^1_2\wedge\sigma^2_1,
\end{aligned}
$$

where $E = \frac{1}{3}\,(9\,u_x + 4\,k + 5)\,(2\,k^2 + 5\,k + 2)^{-1/2}$. This function is an invariant of the pseudo-group. Since $dE = 3\,\sigma^1_2$, it follows that all the derived invariants are constants, and the classifying manifold of the pseudo-group, i.e., the manifold parameterized by differential invariants, is a line. Thus for every two equations (5) such that $(k+2)\,(k+2^{-1}) \neq 0$ the classifying manifolds of their symmetry pseudo-groups (locally) overlap. So these equations are equivalent under a contact transformation, [17, th 15.12]. For example, we can take $k=0$ for one of the equations.

The knowledge of invariant 1-forms defining the symmetry pseudo-groups allows us to find the transformations explicitly.

Theorem 1. *If $k=-2$, then the transformation*

$$
\begin{aligned}
&\tilde{t} = t, \quad \tilde{x} = \exp\left(\frac{4}{3}\,t\right)\left(x + \frac{1}{6}\,y^2\right), \quad \tilde{y} = \exp\left(\frac{2}{3}\,t\right)y,\\
&\tilde{u} = \exp\left(\frac{4}{3}\,t\right)\left(u + \frac{1}{3}\,x - \frac{1}{18}\,y^2\right), \quad \tilde{v} = \mathrm{e}^{2\,t}\left(v - \frac{1}{3}\,y\,u - \frac{2}{9}\,x\,y + \frac{2}{81}\,y^3\right)
\end{aligned}
$$

maps equation (4) into equation (5) written in the tilded variables. In the case $k = -\frac{1}{2}$ the transformation between (4) and (5) has the form

$$
\begin{aligned}
&\tilde{t} = t, \quad \tilde{x} = \exp\left(\frac{2}{3}\,t\right)\left(x + \frac{1}{12}\,y^2\right), \quad \tilde{y} = \exp\left(\frac{1}{3}\,t\right)y,\\
&\tilde{u} = \exp\left(\frac{2}{3}\,t\right)\left(u - \frac{1}{3}\,x - \frac{1}{18}\,y^2\right), \quad \tilde{v} = \mathrm{e}^{t}\left(v - \frac{1}{6}\,y\,u - \frac{1}{18}\,x\,y + \frac{1}{324}\,y^3\right).
\end{aligned}
$$

The transformation

$$
\begin{aligned}
&\tilde{t} = \frac{(3\,\lambda+1)\,(3\,\lambda+2)}{2\,(2\,\lambda+1)}\,t, \quad \tilde{x} = \frac{(2\,\lambda+1)\,\mathrm{e}^{2\lambda t}}{2\,(3\lambda+1)(3\lambda+2)}\,(4\,x + \lambda^2\,y), \quad \tilde{y} = \mathrm{e}^{\lambda t}\,y,\\
&\tilde{u} = \frac{(2\lambda+1)\,\mathrm{e}^{2\lambda t}}{2(3\lambda+1)^2(3\lambda+2)^2}\left(8(2\lambda+1)u - \lambda(\lambda+1)(4x - (5\lambda+2)y^2)\right),\\
&\tilde{v} = \frac{(2\lambda+1)^2\mathrm{e}^{3\lambda t}}{(3\lambda+1)^3(3\lambda+2)^2}\left(24v - 12\lambda(2\lambda+1)(u + (\lambda+1)x) - \lambda^2(\lambda+1)(5\lambda+2)y^3\right),
\end{aligned}
$$

where $\lambda = \frac{2}{9}\left(((k+2)(k+2^{-1}))^{1/2}+k-1\right)$, *maps equation (5) with* $k \notin \{-2,-2^{-1}\}$ *into this equation with* $k=0$ *(written in tilded variables).*

4 Equivalence of linear parabolic equations

The similar but more complicate analysis gives the solution of the contact equivalence problem for the class of linear parabolic equations (LPEs), [11, V. 3, p. 492-523],[19],

$$u_{xx} = T(t,x)\,u_t + X(t,x)\,u_x + U(t,x)\,u. \tag{7}$$

All these equations have infinite-dimensional symmetry pseudo-groups, whose structure depends on the coefficients of equations. For simplicity of notation we consider the following normal form $u_{xx} = u_t + H(t,x)\,u$, while the results are valid for the original equations as well. The solution of the equivalence problem in terms of the coefficients of equations (7) is given in [15].

Define the functions

$$\begin{aligned}
&I = -(H_{xxx})^{1/5}, \quad J_1 = -\frac{2}{5}\,H_{xxxx}\,(H_{xxx})^{-6/5},\\
&J_2 = \frac{1}{2}\left(2\,I\,(J_{1,x}\,J_{1,tx} - J_{1,t}\,J_{1,xx}) - 2\,I_t\,J_{1,x}^2 + I^2\,J_1\,J_{1,t}\,J_{1,x}\right)\,I^{-3}\,J_{1,x}^{-1},\\
&J_3 = -\frac{1}{8}\left(I^4 J_1^2 J_{1,t}^2 - 4 I I_{tt} J_{1,x}^2 - 8 I^2 J_{1,x}^2 H_{xx} + 2 I^3 J_{1,x} J_{1,t}^2 + 8 I_t^2 J_{1,x}^2 - 2 I^3 J_1 J_{1,t} J_{1,tx}\right.\\
&\qquad \left. -2 I^2 J_1 I_t J_{1,t} J_{1,x} + 2 I^3 J_1 J_{1,x} J_{1,tt} - 2 I^5 J_1 J_2 J_{1,t} J_{1,x}\right)\,I^{-6}\,J_{1,x}^{-2},\\
&J_4 = \left(2 I I_{tt} J_{1,x} - 4 I^6 J_3 J_{1,x} + 4 I^2 J_{1,x} H_{xx} - I^3 J_{1,t} + 2 I^3 J_1 J_{1,x} H_x - 4 I_t^2 J_{1,x}\right)\,I^{-6}\,J_1^{-1}\,J_{1,x}^{-1},\\
&L_0 = \left(2 I^2 H_{xx} + I^3 J_1 H_x + I I_{tt} - 2 I_t^2\right)\,I^{-2} J_{1,t}^{-1}, \quad L_1 = (L_{0,x} - I_t)\,I^{-3},\\
&L_2 = \frac{1}{2}\left(2 I L_{0,t} + I J_1 L_0^2 + 4 I^2 H_x - 2 I^3 L_0 L_1 - 4 L_0 I_t\right)\,I^{-5},\\
&M_0 = -2\,M_{1,t}\,\left(3 J_1 M_1 + 2\right)^{-2}, \quad M_1 = \frac{1}{2}\left(I I_{tt} + 2 I H_{xx} + I^3 J_1 H_x - 2 I_t^2\right)\,I^{-6},\\
&M_2 = (M_{0,x} - I_t)\,I^{-3}, \quad M_3 = \frac{1}{8}\left(2 I M_{0,t} - 2 I^3 M_0 M_2 - 4 M_0 I_t + I J_1 M_0^2 + 4 I^2 H_x\right)\,I^{-5}.
\end{aligned}$$

Then we have

Theorem 2. *The class of LPEs is divided into the five subclasses* $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_5$ *invariant under an action of the pseudo-group of contact transformations:*

$\mathcal{P}_1$ *consists of all LPEs such that* $I = 0$*;*

$\mathcal{P}_2$ *consists of all LPES such that* $I \neq 0$ *and* $J_{1x} \neq 0$*;*

$\mathcal{P}_3$ *consists of all LPEs such that* $I \neq 0$, $J_{1x} = 0$, *and* $J_{1t} \neq 0$*;*

$\mathcal{P}_4$ *consists of all LPEs such that* $I \neq 0$, $J_1 = m = const$, *and* $3\,m\,M_1 \neq -2$*;*

$\mathcal{P}_5$ *consists of all LPEs such that* $I \neq 0$, $J_1 = m = const$, *and* $3\,m\,M_1 = -2$.

Every equation from the subclass $\mathcal{P}_1$ *is equivalent to the linear heat equation* $u_{xx} = u_t$.

Two equations from $\mathcal{P}_2$ *are locally equivalent to each other if and only if they have the same functional dependencies among the differential invariants* J_1, J_2, J_3, J_4, *and their invariant derivatives.*

Two equations from $\mathcal{P}_3$ *are locally equivalent to each other if and only if they have the same functional dependencies among the differential invariants* J_1, L_1, L_2, *and their invariant derivatives.*

Two equations from $\mathcal{P}_4$ *are locally equivalent to each other if and only if they have the same functional dependencies among the differential invariants* M_1, M_2, M_3, *and their invariant derivatives.*

Every equation from the subclass $\mathcal{P}_5$ *is locally equivalent to the equation* $u_{xx} = u_t + \tilde{m}\,x^{-2}\,u$ *provided* $\tilde{m} = -4/(3\,m^5)$.

The expression of the function I in terms of the coefficients of equations (7) and the neccessary and sufficient condition for equivalence to the linear heat equation are found in [10]. Some particular cases of invariants — so-called Laplace type invariants — are found in [9]. Numerous examples of equations (7) from the first subclass are given in [10, 22].

5 Equivalence of nonlinear wave equations

We outline the solution of the contact equivalence problem for the following class of nonlinear wave equations

$$u_t = a(x,u)\,v_x, \qquad v_t = b(x,u)\,u_x. \tag{8}$$

As it is shown in [8], these equations are equivalent to the class of equations $w_{tt} = f(x,w_x)\,w_{xx} + g(x,w_x)$ studied in [7]. In [8], the symmetry classification for systems (8) is given in the finite-dimensional cases. In [16], the moving coframe method is applied in both finite-dimensional and infinite-dimensional cases. The results are gathered in the following theorem.

Theorem 3. *Every system from the class of nonlinear wave equations (8) is equivalent under a contact transformation to a system from one of the five invariant subclasses* $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$, *and* $\mathcal{Q}_5$*:*

$\mathcal{Q}_1$ *consists of all systems (8) such that* $(b/a)_x \neq 0$, $(a\,b)_u \neq 0$, *and* $b^{-1}\,(b/a)_x\,(b\,a)_u \neq const$*;*

$\mathcal{Q}_2$ *consists of all systems* $u_t = a(x,u)\,v_x$, $v_t = a(x,u)\,u_x$ *such that* $(\ln a)_{xu} \neq 0$*;*

$\mathcal{Q}_3$ *consists of all systems* $u_t = a(u)\,v_x$, $v_t = a(u)\,u_x$ *such that*
$N_1 = a\,\left(a_u^2 a_{uu} + a a_{uu}^2 - a a_u a_{uuu}\right)\,a_u^{-4} \neq const$*;*

$\mathcal{Q}_4$ *consists of all systems* $u_t = a(u)\,v_x$, $v_t = a(u)\,u_x$ *such that* $N_1 = const$*;*

$\mathcal{Q}_5$ *consists of the system* $u_t = v_x$, $v_t = u_x$*.*

For systems from subclasses $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *the symmetry groups are finite-dimensional, and their dimensions are between 2 and 6.*

For systems from subclasses $\mathcal{Q}_3$, $\mathcal{Q}_4$, *and* $\mathcal{Q}_5$ *the symmetry pseudo-groups are infinite-dimensional. The equations from* $\mathcal{Q}_3$ *and* $\mathcal{Q}_4$ *are linearizable by a hodograph transformation. Two equations from* $\mathcal{Q}_3$ *are equivalent under a contact transformation iff they have the same functional dependencies among the differential invariants* N_1, $N_2 = 2N_{1,u}^{-1}(a_{uu}a_u^{-1} - a_u a^{-1}) + (N_{1,u}^{-1})_u$, *and* $N_3 = \left(4a^2 N_{1,u}^2 a_u^{-2} - 1\right) N_2 + 4N_{1,u}^{-1}\left(a^2 N_{1,u} a_u^{-2}\right)_u$.

Two systems from the subclass $\mathcal{Q}_4$ *are equivalent if and only if they have the same constant value of the invariant* N_1.

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